

LECTURE #2: THE QUBIT

Date: 9/5/25

The goal of this lecture is to understand what makes quantum bits different from classical bits.

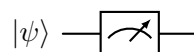
Classical bits vs. qubits. As it turns out, classical bits (or, the digital bits that we use in classical digital computers) and quantum bits (or, the fundamental unit of information we use in quantum computation) are fundamentally different. Let us take a closer look at how they are represented.

	Classical Bits	Quantum Bits (Qubits)
State space	System with two logical states: "0" or "1", represented in binary.	A quantum mechanical two-level system: a qubit is a complex linear combination $ \psi\rangle = \alpha 0\rangle + \beta 1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $ \alpha ^2 + \beta ^2 = 1$. Basis states are vectors in $\mathbb{C}^2$, e.g. $ 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Physical realization	Transistors / digital circuits: either low-voltage with $\approx 0V$ ("0") or high-voltage $\approx 5V$ ("1").	Electron spin, photon polarization, trapped ions, superconducting circuits, etc.
Measurement	Deterministic read-out: outcomes are "0" or "1" (e.g., voltmeter).	Probabilistic: collapses to $ 0\rangle$ or $ 1\rangle$ with probabilities $ \alpha ^2, \beta ^2$.

Table 1: Comparison between classical bits and quantum bits.

Important note about observing qubits. Reading a qubit is an inherently *destructive process*—once measured, the state of the qubit is, in general, irreversibly changed. We say that the state *collapses* into a definitive state, effectively becoming a classical bit.

A quantum measurement of a qubit $|\psi\rangle$ is denoted using the following wire notation:



The symbol on the far right represents a quantum measurement apparatus (e.g., similar to voltmeter in digital circuits); this represents the instrument used to carry out the measurement.

The probabilities of observing the outcomes $|0\rangle$ and $|1\rangle$ as results of a measurement are as follows:

$$|\psi\rangle \text{ --- } \boxed{\text{meter}} = \begin{cases} |0\rangle & \text{with probability } |\alpha|^2 \\ |1\rangle & \text{with probability } |\beta|^2 \end{cases}$$

This is called the *Born rule* of quantum mechanics.

If the qubit is measured a second time, the same value will appear as the first time it was measured—it becomes a classical bit. Formally, in circuit notation, this means that

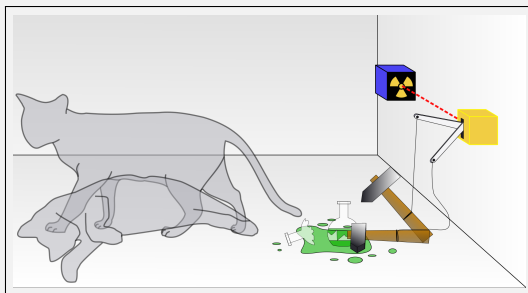
$$|\psi\rangle \text{ --- } \boxed{\text{meter}} \text{ --- } \equiv |\psi\rangle \text{ --- } \boxed{\text{meter}} \text{ --- } \boxed{\text{meter}} \text{ --- }$$

The qubit is now set to the value it had at the time of the first measurement and will not change.

Example: Schrödinger's cat state.

An iconic example of a qubit is the state $|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$.

In 1935, Schrödinger proposed the following thought experiment: a cat is put in a box with a cesium atom in an excited state corresponding to $|1\rangle$. At any time, the atom can spontaneously decay towards $|0\rangle$, thereby producing radiation which sets off a detector that releases poison, and kills the cat. The atom is a microscopic object and obeys the rules of quantum mechanics; for the sake of the thought experiment, we will assume that its state is precisely given by an equal superposition of $|0\rangle$ and $|1\rangle$.



The question is: is the cat dead or alive? We don't know until the box is opened. If we accept the laws of quantum mechanics, we are led to believe that the cat—a macroscopic object—is in an equal superposition of being dead and alive, which we can map onto the measurement outcomes $|0\rangle$ or $|1\rangle$.

Relative phases. Another example of a qubit is the quantum state given by

$$|\phi\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle.$$

At first sight, this state seems identical to the one at the beginning of this section, except for the (-1) sign instead of a $(+1)$ sign. This (-1) sign is called a *relative phase*, since it is affecting only one of the amplitudes—as it turns out, this has important consequences even though it has no observable effect on the probabilities of each outcome. Still, we will see shortly that $|\psi\rangle$ and $|\phi\rangle$ are not the same state!

Global phases. What about $|\psi\rangle$ and $-|\psi\rangle$? If the phase is applied to the entire state, it turns out not to be possible to detect any differences between $|\psi\rangle$ and $-|\psi\rangle$. In other words, one cannot be distinguished from the other. Therefore, $-$ has no effect in a global context.

Going forward, $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ represent the same state, for any angle $\phi \in [0, 2\pi]$. We say that the complex number $e^{i\phi}$ with modulus $|e^{i\phi}| = 1$ denotes a global phase.

How to represent a general qubit? Any qubit can be written as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle$$

for $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. Using this representation, we can now visualize a qubit in the so-called Bloch sphere, shown in Fig. 1. The z axis points up. The x axis points to the right. The y axis points away from the figure and into the distance, thereby contributing to the sphere's 3D appearance. Note that the points for $|0\rangle$ and $|1\rangle$ are precisely at the top and bottom of the sphere, respectively. This is a little counterintuitive, as the states $|0\rangle$ and $|1\rangle$ are orthogonal as vectors in the two-dimensional complex vector space $\mathbb{C}^2$. Classical bits can thus be represented on the Bloch sphere, but they are merely limited to the north and south pole.

Note the placement of the angles for θ and φ in Fig. 1. All states are unit vectors which lie on the surface. While the information stored in θ and φ is continuous and may carry an infinite amount of precision, the information is not easily accessible, and may be lost as the result of a measurement. For example, if $\theta = 0.11010$, a measurement may result in $|0\rangle$, and the information stored in the fractional part of θ is lost.

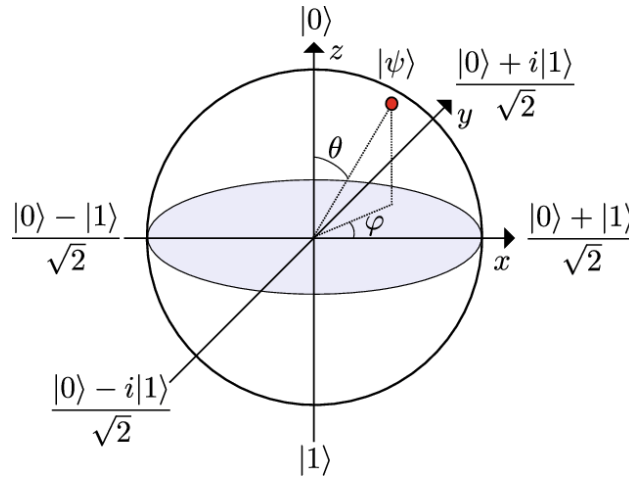


Figure 1: The Bloch sphere.

Quantum information processing: How does a qubit evolve in a quantum computation?

1. Through linear transformations represented by complex matrices.
2. Through measurement, usually at the end of the computation.

Single-qubit transformations: In the following example, we fix a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, for some $\alpha, \beta \in \mathbb{C}$, and see how it gets transformed under various linear transformations.

We summarize some of the most frequently used single-qubit gates. Note that each gate is a *linear transformation*, and thus can be specified by its action on the computational basis states $|0\rangle$ and $|1\rangle$.

1. **Identity Gate (I)** The identity operation leaves the state unchanged. Although trivial in concept, it is paradoxically nontrivial to implement in practice due to noise.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I|0\rangle = |0\rangle, \quad I|1\rangle = |1\rangle, \quad I|\psi\rangle = |\psi\rangle.$$

2. **Pauli-X Gate (Bit-Flip)** Analogous to the classical NOT gate, it swaps $|0\rangle$ and $|1\rangle$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = |1\rangle,$$

$$X|1\rangle = |0\rangle,$$

$$X(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle + \alpha|1\rangle.$$

3. **Pauli-Y Gate** Introduces a bit- and phase-flip with imaginary factors.

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|0\rangle = i|1\rangle,$$

$$Y|1\rangle = -i|0\rangle,$$

$$Y(\alpha|0\rangle + \beta|1\rangle) = -i\beta|0\rangle + i\alpha|1\rangle.$$

4. **Pauli-Z Gate (Phase-Flip)** Leaves $|0\rangle$ unchanged while flipping the phase of $|1\rangle$.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|0\rangle = |0\rangle,$$

$$Z|1\rangle = -|1\rangle,$$

$$Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle.$$

5. **Hadamard Gate (H)** Creates equal superpositions and is fundamental for quantum algorithms.

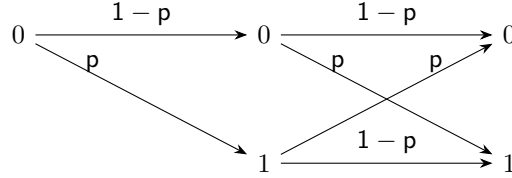
$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Note that I, X, Y and Z are the famous *Pauli matrices*. Only X and I have classical analogues; Y and Z have no counterpart in the classical setting.

Why are qubits not just probabilistic bits? This is clearly something the reader might be wondering at this point. To illustrate why the two concepts are different, let's conduct two experiments.

Experiment 1: This is a *probabilistic computation* with a “probabilistic bit”. Let $p \in (0, 1)$. With a probability of p , the bit will flip, and with a probability of $1 - p$, the bit will stay the same. Here's a Markov transition diagram for two steps of a simple randomized process in which the initial bit that starts out as "0":



After two steps, this yields a probabilistic bit $b \in \{0, 1\}$. Concretely, we can calculate that

$$\Pr[b = 0] = (1 - p) \cdot (1 - p) + p^2 = 1 - 2p + 2p^2$$

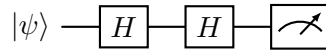
$$\Pr[b = 1] = p \cdot (1 - p) + p \cdot (1 - p) = 2p - 2p^2.$$

If we fix $p = \frac{1}{2}$, then we find that

$$\Pr[b = 0] = \frac{1}{2} \quad \text{and} \quad \Pr[b = 1] = \frac{1}{2}.$$

This situation can be equated to the probabilistic situation of flipping a random coin.

Experiment 2: Perform a similar random process on a quantum bit. Start in $|0\rangle$ and apply the Hadamard gate to it twice to introduce randomness, and then measure the final output. Concretely, we consider:



The first application of the Hadamard gate H results in the state

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

By linearity, the second application of the Hadamard gate H results in

$$\begin{aligned}
 \frac{1}{\sqrt{2}}H|0\rangle + \frac{1}{\sqrt{2}}H|1\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) \\
 &= \frac{1}{2}(|0\rangle + |1\rangle) + \frac{1}{2}(|0\rangle - |1\rangle) \\
 &= \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle \\
 &= \left(\frac{1}{2} + \frac{1}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{1}{2}\right)|1\rangle \\
 &= 1 \cdot |0\rangle + 0 \cdot |1\rangle \\
 &= |0\rangle.
 \end{aligned}$$

Therefore, the outcome of the experiment is *deterministic*, and always yields $|0\rangle$ with probability 1. This is not what we would've expected! In particular, it's not probabilistic behavior. What happened?

Quantum interference! It turns out that quantum mechanical amplitudes are more like *probability waves* which have wave-like properties. The amplitude of one application of H “cancel out” the amplitude of the other application of H , leading to a definitive and deterministic answer.

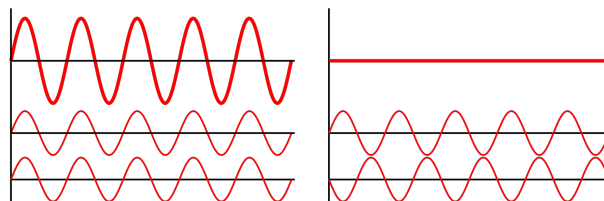


Figure 2: Constructive and destructive interference.

Sneak peek for Lecture 3: multiple qubits. What happens when we consider a quantum state on more than just one qubit. We will see that the number of amplitudes scales exponentially in the number of qubits:

1 qubit: A general single-qubit state has two amplitudes and is of the form:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle .$$

2 qubits: A general two-qubit state has four amplitudes and is of the form:

$$|\psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle .$$

⋮

n qubits: A general n -qubit state has 2^n amplitudes and is of the form

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle .$$