CS 599 P1: Introduction to Quantum Computation — Boston University, Fall 2025
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LECTURE #4: MEASUREMENTS & HEISENBERG UNCERTAINTY PRINCIPLE

Date: 9/11/25

The goal of this lecture is to explore the different ways qubits can be measured, and how measurement on multiple systems differs from that on a single system. We begin by reviewing the standard procedure for measuring a single qubit, then introduce a new perspective by describing measurement in terms of linear operators. Next, we examine measurements in different bases, showing how each basis defines a distinct set of states into which the qubit may collapse, leading to different possible outcomes. We then formally define quantum measurement using the *fourth axiom of quantum mechanics*, which states that measurements are represented by a collection of measurement operators acting on the state of a system. The lecture also addresses the fundamental limitations of measurement, as illustrated by the *Heisenberg uncertainty principle*, which highlights the impossibility of carrying out certain quantum measurements at the same time. Finally, we extend the discussion from single systems to composite systems, demonstrating how measurement operators can be applied across multiple qubits.

1 Review: Quantum measurement

Previously, we learned that a measurement of a qubit is destructive and occurs in a probabilistic manner according to *Born's rule*. Suppose the state of our system is given by

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\ ,$$

where $|\alpha_1|^2 + |\beta|^2 = 1$. Then, our measurement reveals an outcome $|0\rangle$ or $|1\rangle$ such that:

$$\Qcircuit @C=1em @R=.7em { \lstick{|\psi\rangle} & \meter } \quad = \begin{cases} |0\rangle \ \text{ with probability } \ |\alpha|^2 \\ |1\rangle \ \text{ with probability } \ |\beta|^2. \end{cases}$$

2 Measurement operators

Let us now consider an alternative perspective on the Born rule from above; specifically, using the language of *measurement operators*. Suppose that we have a single-qubit system which lives in a 2-dimensional complex vector space $\mathbb{C}^2$ and which comes with an orthonormal basis $\{|0\rangle\,, |1\rangle\}$.

Previously, we encountered the so-called identity operation; this is the quantum gate of the form

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As you will show in Homework #1, Problem 4, we can conveniently characterizes the identity operation via its action on the computational basis $\{|0\rangle, |1\rangle\}$ and write

$$I = |0\rangle\langle 0| + |1\rangle\langle 1| \qquad \text{(resolution of the identity)}$$

So what exactly are the operators $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$? First note that they are *linear* operators; in particular, they map vectors to vectors. In matrix form, these are given by the matrices

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

You should convince yourself that they sum to I, and that they act as desired on the basis $\{|0\rangle, |1\rangle\}$. Do $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ represent quantum gates? Not exactly: despite the fact that they represent linear operators, they are not in fact *unitary matrices* and thus do not comprise valid quantum gates. Instead, these matrices form so-called *orthogonal projectors*, which are convenient mathematical tools. Letting

$$\Pi_0 := |0\rangle\langle 0| \quad \text{and} \quad \Pi_1 := |1\rangle\langle 1|$$

we find that $\Pi_0^2 = \Pi_0$ and $\Pi_1^2 = \Pi_1$; meaning, if we apply these operators twice, it's as if we had applied them once: they *project* down to the subspaces $\mathrm{span}\{|0\rangle\}$ and $\mathrm{span}\{|1\rangle\}$ such that

- Π_0 projects onto the 1-dimensional subspace $\mathrm{span}\{|0\rangle\} = \mathrm{span}\left\{\lambda \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,:\, \lambda \in \mathbb{C}\right\}$
- Π_1 projects onto the 1-dimensional subspace $\mathrm{span}\{|1\rangle\} = \mathrm{span}\left\{\lambda \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,:\, \lambda \in \mathbb{C}\right\}$

Moreover, because $\{|0\rangle, |1\rangle\}$ is an orthonormal basis, they are also *orthogonal* in the sense that

$$\Pi_0\Pi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \Pi_1\Pi_0.$$

More generally, we define orthogonal projectors as follows.

Definition: Orthogonal projectors

An ensemble of linear operators $\{\Pi_i\}_{i\in[m]}$ with $[m] = \{1, \dots, m\}$ consists of *orthogonal projectors*, if

- $\Pi_i\Pi_j = \delta_{ij} \cdot \Pi_i, \quad \forall i, j \in [m]$
- $(\Pi_i)^\dagger = \Pi_i, \quad \forall i \in [m].$

where δ_{ij} is the Kronecker delta which is equal to 1, if $i = j$, and equal to 0, whenever $i \neq j$.

We will now see why the set $\{\Pi_0, \Pi_1\}$ from before can be thought of as *measurement operators*.

Revisiting the Born rule. We will now use the machinery of orthogonal projectors to describe the notion of a quantum measurement. The high level idea is that the projectors $\{\Pi_0, \Pi_1\}$ have the ability to break any quantum state down into 1-dimensional subspaces which each carry an amplitude. The question of "what is the amplitude corresponding to the outcome $|0\rangle$ or $|1\rangle$?" then translates into "what is the amplitude on a given subspace?" Generally speaking, for any qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha, \beta \in \mathbb{C}$, we can write

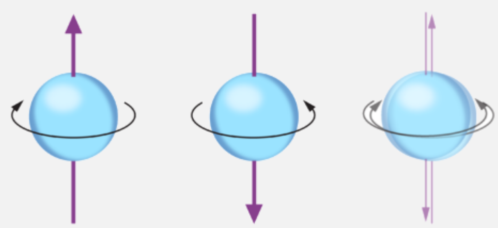
$$\begin{aligned}
|\psi\rangle &= I \cdot |\psi\rangle \\
&= (\Pi_0 + \Pi_1) \cdot |\psi\rangle && \text{(resolution of the identity)} \\
&= \Pi_0 |\psi\rangle + \Pi_1 |\psi\rangle && \text{(by linearity)} \\
&= \underbrace{(|0\rangle\langle 0|) \cdot |\psi\rangle}_{\text{how much of } |0\rangle \text{ is contained in } |\psi\rangle} + \underbrace{(|1\rangle\langle 1|) \cdot |\psi\rangle}_{\text{how much of } |1\rangle \text{ is contained in } |\psi\rangle} \\
&= \langle 0|\psi\rangle \cdot |0\rangle + \langle 1|\psi\rangle \cdot |1\rangle \\
&= \langle 0| \big(\alpha|0\rangle + \beta|1\rangle\big) \cdot |0\rangle + \langle 1| \big(\alpha|0\rangle + \beta|1\rangle\big) \cdot |1\rangle \\
&= \alpha|0\rangle + \beta|1\rangle .
\end{aligned}$$

where $\langle 0|\psi\rangle = \alpha$ and $\langle 1|\psi\rangle = \beta$ are the inner products between $|\psi\rangle$ and the basis vectors $\{|0\rangle, |1\rangle\}$, and where we used the orthonormality condition of the basis; meaning $\langle i|j\rangle = \delta_{i,j}$ for $i, j \in \{0, 1\}$.

What does the above measurement correspond to physically?

Example: Spin-alignment on the z-axis.

Suppose that the quantum state of our qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ describes the *spin* of an electron; meaning, it is in a superposition of two states given by spin-up and spin-down.



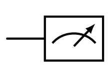
Measuring in the $\{|0\rangle, |1\rangle\}$ basis tells us how the spin is aligned with respect to the z-axis of the Bloch sphere: if the outcome is $|0\rangle$, the spin points up, whereas if the outcome is $|1\rangle$, the spin points down.[a]

[a]Image credit: https://blog.ifs.com/a-quantum-leap-in-computing-power/

3 Measurement in a different basis

It turns out that, in quantum mechanics, there is actually nothing special about the computational basis $\{|0\rangle, |1\rangle\}$—in fact, we can just as well measure in any other basis of $\mathbb{C}^2$. We will see that measurement in one basis basis can provide us with completely different information relative to another basis.

Suppose $\{|b_0\rangle, |b_1\rangle\}$ is another orthonormal basis of $\mathbb{C}^2$. Once again, we can ask:

$|\psi\rangle$  $=$ is $|\psi\rangle$ either $|b_0\rangle$ or is it $|b_1\rangle$?

Not surprisingly, we can once again use the resolution of the identity, and write

$$I = |b_0\rangle\langle b_0| + |b_1\rangle\langle b_1|.$$

As before, we can introduce the projectors

$$\Pi_0 := |b_0\rangle\langle b_0| \quad \text{and} \quad \Pi_1 := |b_1\rangle\langle b_1|$$

Because $\{|b_0\rangle, |b_1\rangle\}$ is also an orthonormal basis of $\mathbb{C}^2$, we find again that $\Pi_0^2 = \Pi_0$ and $\Pi_1^2 = \Pi_1$; meaning, if we apply these operators twice, it's as if we had applied them once: they *project* down to the subspaces $\mathrm{span}\{|b_0\rangle\}$ and $\mathrm{span}\{|b_1\rangle\}$ such that

- Π_0 projects onto the 1-dimensional subspace $\mathrm{span}\{|b_0\rangle\}$
- Π_1 projects onto the 1-dimensional subspace $\mathrm{span}\{|b_1\rangle\}$

Moreover, because $\{|b_0\rangle, |b_1\rangle\}$ is an orthonormal basis, Π_0 and Π_1 are also *orthogonal* in the sense that

$$\Pi_0\Pi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \Pi_1\Pi_0.$$

Now, suppose we have a qubit $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ with $\alpha, \beta \in \mathbb{C}$ which is described to us in the basis $\{|0\rangle, |1\rangle\}$. Because $\{|b_0\rangle, |b_1\rangle\}$ is also an orthonormal basis of $\mathbb{C}^2$, we can express the basis vectors $|0\rangle$ and $|1\rangle$ each as a linear combination of the new basis vectors $\{|b_0\rangle, |b_1\rangle\}$. Plugging this in, we can write

$$|\psi\rangle = \gamma |b_0\rangle + \delta |b_1\rangle$$

for some new coefficients $\gamma, \delta \in \mathbb{C}$ such that $|\gamma|^2 + |\delta|^2 = 1$. Then, we can once again write

$$\begin{aligned}
|\psi\rangle &= I \cdot |\psi\rangle \\
&= (\Pi_0 + \Pi_1) \cdot |\psi\rangle && \text{(resolution of the identity)} \\
&= \Pi_0 |\psi\rangle + \Pi_1 |\psi\rangle && \text{(by linearity)} \\
&= \underbrace{(|b_0\rangle\langle b_0|) \cdot |\psi\rangle}_{\text{how much of } |b_0\rangle \text{ is contained in } |\psi\rangle} + \underbrace{(|b_1\rangle\langle b_1|) \cdot |\psi\rangle}_{\text{how much of } |b_1\rangle \text{ is contained in } |\psi\rangle} \\
&= \langle b_0|\psi\rangle \cdot |b_0\rangle + \langle b_1|\psi\rangle \cdot |b_1\rangle \\
&= \langle b_0| \big(\gamma |b_0\rangle + \delta |b_1\rangle\big) \cdot |b_0\rangle + \langle b_1| \big(\gamma |b_0\rangle + \beta |b_1\rangle\big) \cdot |b_1\rangle \\
&= \gamma |b_0\rangle + \delta |b_1\rangle .
\end{aligned}$$

where we used the orthonormality condition of the basis; meaning $\langle b_i|b_j\rangle = \delta_{i,j}$ for $i, j \in \{0, 1\}$.

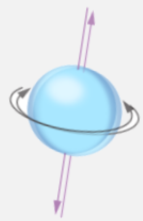
An example of a measurement in a basis which is not in the computational basis is the so-called *Hadamard basis* measurement (see Homework #, Problem 2.3). Here, the measurement basis is specified by another orthonormal basis $\{|+\rangle, |-\rangle\}$ of $\mathbb{C}^2$, where

$$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

What does the above measurement correspond to physically?

Example: Spin-alignment on the x-axis.

Suppose that the quantum state of our qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ describes the *spin* of an electron; meaning, it is in a superposition of two states given by spin-up and spin-down.



Measuring in the $\{|+\rangle, |-\rangle\}$ basis tells us how the spin is aligned with respect to the x-axis of the Bloch sphere: if the outcome is $|+\rangle$, the spin is horizontally aligned with the "+" side of the x-axis, whereas if the outcome is $|-\rangle$, the spin is horizontally aligned with the "−" side of the x-axis.[a]

[a] Image credit: https://blog.ifs.com/a-quantum-leap-in-computing-power/

4 Measurement postulate of quantum mechanics

We will now formally state the so-called measurement postulate of quantum mechanics. We will use the *Von Neumann measurement*, which is also called a projective measurement. This is the standard measurement framework introduced by John von Neumann in the early formulation of quantum mechanics, and which encompasses virtually all measurements you will ever encounter in practice.[1]

Axiom 4 (Measurement)

A measurement is a process in which information about a physical system is acquired by an observer. In quantum mechanics, measurements are described by a collection of measurement operators $\{\Pi_i\}$ acting on the state of a system; these are orthogonal projectors such that $\sum_i \Pi_i = I$.

If the system is in the state $|\psi\rangle$, the probability that we observe the label i is given by

$$\begin{aligned}\Pr\left[\boxed{} = \text{"outcome } i\text{"} \right] &= \| \Pi_i |\psi\rangle \|^2 \\ &= \langle\psi| \Pi_i^\dagger \Pi_i |\psi\rangle \qquad \text{(since } \{\Pi_i\} \text{ are orthogonal projectors)} \\ &= \langle\psi| \Pi_i |\psi\rangle.\end{aligned}$$

Moreover, the state after the measurement is another normalized state of the form

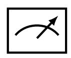
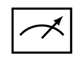
$$\frac{\Pi_i |\psi\rangle}{\sqrt{\langle\psi| \Pi_i |\psi\rangle}}.$$

[1] Note that there are also more general notions of a measurement; however, without loss of generality, these can be described by a projective measurement onto a larger system. This is known as Naimark's dilation theorem.

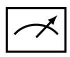
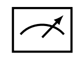
5 Heisenberg Uncertainty Principle

It turns out that quantum mechanics enforces certain fundamental limits on what an observer can infer about the state of a system. The *Heisenberg uncertainty principle* states that certain pairs of observables cannot be simultaneously measured with arbitrary precision. For qubits, this principle can be illustrated using two complementary bases: the computational basis $\{|0\rangle, |1\rangle\}$ and the Hadamard basis $\{|+\rangle, |-\rangle\}$.

Suppose that a qubit is prepared in the state $|0\rangle$. If we measure the qubit in the computational basis $\{|0\rangle, |1\rangle\}$, the outcome is *certain* in the sense that

$\Pr\Big[$  $= \text{"0"}\Big] = 1$ and $\Pr\Big[$  $= \text{"1"}\Big] = 0.$

However, if we measure the same state $|0\rangle$ in the Hadamard basis, the outcomes become completely random:

$\Pr\Big[$  $= \text{"+"}\Big] = |\langle +|0\rangle|^2 = \frac{1}{2}$ and $\Pr\Big[$  $= \text{"-"}\Big] = |\langle -|0\rangle|^2 = \frac{1}{2}.$

This loosely illustrates the uncertainty principle: knowing the state exactly in one basis (say, the computational basis) implies complete uncertainty in the complementary basis (the Hadamard basis). In this case, we say that the the two measurement bases are mutually incompatible.

6 Measurements on Multiple Systems

Finally, we describe how the same formalism of projective measurements also applies to multiple quantum systems. Suppose now that we have two Hilbert spaces, say $\mathcal{H}_A$ and $\mathcal{H}_B$, and we wish to describe quantum measurements on the joint system given by $\mathcal{H}_A \otimes \mathcal{H}_B$.

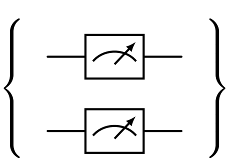
Suppose $\mathcal{H}_A$ has dimension d_A with orthonormal basis $\{|a_i\rangle\}_{i=0,\ldots,d_A-1}$ and $\mathcal{H}_B$ has dimension d_B with orthonormal basis $\{|b_j\rangle\}_{j=0,\ldots,d_B-1}$. Then the basis of the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ is

$$\left\{ |a_i\rangle_A \otimes |b_j\rangle_B \;\middle|\; i = 0, \ldots, d_A - 1,\ j = 0, \ldots, d_B - 1 \right\}.$$

Suppose also that we have an arbitrary joint state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ which can be written as

$$|\psi\rangle_{AB} = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} \alpha_{ij}\ |a_i\rangle_A \otimes |b_j\rangle_B\,.$$

To conduct a measurement on multiple systems, we only need to slightly extend the formalism from the previous section. Concretely, measuring $|\psi\rangle_{AB}$ in the basis above yields

$|\psi\rangle_{AB}$  yields outcome label (i, j) with probability $\|\Pi_{ij}\,|\psi\rangle_{AB}\|^2$

where we introduce the orthogonal projector on system AB of the form

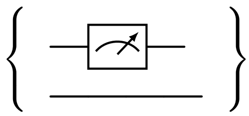
$$\Pi_{ij} = |a_i\rangle\langle a_i|_A \otimes |b_j\rangle\langle b_j|_B.$$

For example, if $\mathcal{H}_A$ uses the canonical basis $\{|i\rangle\}_{i=0,\ldots,d_A-1}$ and $\mathcal{H}_B$ uses the canonical basis $\{|j\rangle\}_{j=0,\ldots,d_B-1}$, then the projector takes the simple form given by

$$\Pi_{ij} = |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B.$$

Measurements on a single system. To describe a measurement which ocurrs on only a single system, we simply modify the previous multiple system measurement method as follows.

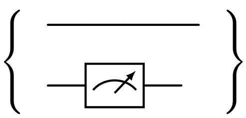
Suppose we only measure system A in the basis $\{|a_i\rangle\}_{i=0,\ldots,d_A-1}$. Then,

$|\psi\rangle_{AB}$  yields outcome label i with probability $\left\|(\Pi_i \otimes I_B)\,|\psi\rangle_{AB}\right\|^2$

where we introduce the orthogonal projector on system A of the form

$$\Pi_i = |a_i\rangle\langle a_i|_A.$$

Similarly, if we only measure system B in the orthonormal basis $\{|b_j\rangle\}_{j=0,\ldots,d_B-1}$. Then,

$|\psi\rangle_{AB}$  yields outcome label j with probability $\left\|(I_A \otimes \Pi_j)\,|\psi\rangle_{AB}\right\|^2$

where we introduce the orthogonal projector on system B of the form

$$\Pi_j = |b_j\rangle\langle b_j|_B.$$