CS 599 P1: Introduction to Quantum Computation — Boston University, Fall 2025
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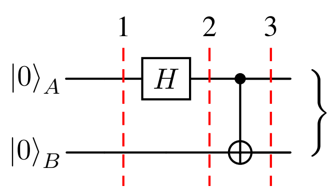
LECTURE #6: QUANTUM ENTANGLEMENT & QUANTUM TELEPORTATION

Date: 9/18/25

In this lecture, we will encounter some of the "spookier" aspects of quantum mechanics; namely, the notion of quantum entanglement. We begin by examining maximally entangled states, or Bell states, and what makes them unique. From there, we briefly touch on the famous Einstein–Podolsky–Rosen (EPR) paradox, which challenged the completeness of quantum theory and introduced the idea of "spooky action at a distance." Finally, we apply these concepts to the remarkable protocol of quantum teleportation, where entanglement, together with classical communication, allows the transfer of an unknown quantum state across space without physically sending the particle itself.

1 The Bell state

Last time, we briefly encountered the following interesting looking two-qubit state, which is prepared by a simple quantum circuit consisting of a Hadamard gate and a CNOT gate:



$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

Let us verify the output of the circuit by keeping track of how the state evolves in each step.

Step 1: Initializing the input in AB. We begin with two qubits, both initialized in the state

$$|0\rangle_A \otimes |0\rangle_B .$$

Step 2: Applying the Hadamard to qubit A. Applying a Hadamard to qubit A creates a superposition:

$$(H_A \otimes I_B)\big(|0\rangle_A \otimes |0\rangle_B\big) = \Big(\tfrac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)\Big) \otimes |0\rangle_B .$$

Expanding, this in the canonical computational basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$, we get

$$\frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |0\rangle_B\big).$$

Step 3: Applying a CNOT with control A and target B. Next, a controlled-NOT gate flips qubit B when qubit A is in state $|1\rangle$. Acting on the state above, we obtain

$$\frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\big).$$

Therefore, the resulting state results exactly in the famous *Bell state*:

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B).$$

Why makes the Bell state special? The reason the Bell state $|\Phi^+\rangle_{AB}$ is a special quantum state is because it is an *entangled state*—it cannot be written as a tensor product of two "local" states.

Definition: Quantum entanglement

A two-qubit quantum state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is called *entangled*, if it cannot be decomposed as a tensor product of two single-qubit states such that

$$|\psi\rangle_{AB} = |\psi_1\rangle_A \otimes |\psi_2\rangle_B \ ,$$

for *any* choice of states $|\psi_1\rangle_A$ and $|\psi_2\rangle_B$. Otherwise, it is called a *product state* or *separable state*.

To illustrate the notion of quantum entanglement, let's look at examples of quantum states where such a decomposition is in fact possible. Consider the two-qubit state

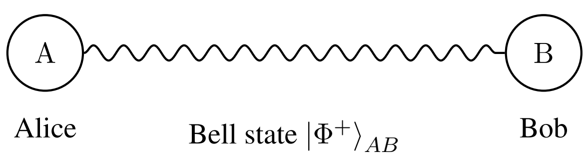
$$|\psi\rangle_{AB} = \tfrac{1}{2}\big(|00\rangle - |01\rangle + |01\rangle - |11\rangle\big).$$

This state is clearly not entangled because

$$|\psi\rangle_{AB} = \left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle_B - |1\rangle_B}{\sqrt{2}}\right).$$

2 Why is Entanglement Spooky?

Quantum entanglement becomes particularly striking when we imagine that two parties, Alice and Bob, generate an entangled pair of qubits and then go their separate ways and travel over large distances.



For instance, suppose Alice takes her qubit A with her on a journey to the Moon, while Bob remains on Earth with the other qubit B. Despite the enormous distance between them, the two qubits remain described by a single joint quantum state $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The question is:

What happens when Alice performs a measurement on her qubit? Will Bob's qubit be affected instantaneously, even across such vast separations?

To explore this, we consider two thought experiments which are characterized by different measurement choices by Alice. As we will see, these reveal the "spooky" nature of quantum correlations.

Experiment 1: Measurement in the Computational Basis. Suppose Alice measures her qubit in the computational basis $\{|0\rangle, |1\rangle\}$. The probability of observing outcome $i \in \{0,1\}$ is given by

$$\Pr\left[\boxed{\nearrow} = |i\rangle\right] = \left\|(|i\rangle\langle i|_A \otimes I_B)\,|\Phi^+\rangle_{AB}\right\|^2,$$

and the corresponding post-measurement state is

$$\frac{(|i\rangle\langle i|_A \otimes I_B)\,|\Phi^+\rangle_{AB}}{\sqrt{\langle\Phi^+|_{AB}\,(|i\rangle\langle i|_A \otimes I_B)\,|\Phi^+\rangle_{AB}}}.$$

• **Outcome “0”:** Alice obtains outcome 0 with probability

$$\left\|(|0\rangle\langle 0|_A \otimes I_B)\,|\Phi^+\rangle_{AB}\right\|^2 = \left\|\tfrac{1}{\sqrt{2}}\,|0\rangle_A \otimes |0\rangle_B\right\|^2 = \tfrac{1}{2}\,\||0\rangle_A \otimes |0\rangle_B\|^2 = \tfrac{1}{2},$$

and the post-measurement state is

$$|0\rangle_A \otimes |0\rangle_B\,.$$

• **Outcome “1”:** Alice obtains outcome 1 with probability

$$\left\|(|1\rangle\langle 1|_A \otimes I_B)\,|\Phi^+\rangle_{AB}\right\|^2 = \left\|\tfrac{1}{\sqrt{2}}\,|1\rangle_A \otimes |1\rangle_B\right\|^2 = \tfrac{1}{2}\,\||1\rangle_A \otimes |1\rangle_B\|^2 = \tfrac{1}{2},$$

and the post-measurement state is

$$|1\rangle_A \otimes |1\rangle_B\,.$$

Therefore, once Alice measures her qubit A, she obtains a random bit; moreover, Bob's qubit in B is *instantaneously* determined to be of the exact same value, even if Bob is far away. *How spooky is this?*

At first glance, this phenomenon may not seem so mysterious. After all, one could imagine a purely classical scenario: before Alice and Bob separate, a random bit is generated and written down on two identical slips of paper. Alice receives one slip in an envelope, Bob the other. When Alice opens her envelope and discovers the bit value, she immediately knows what Bob will find as well. In this sense, the correlations observed in the computational basis could, in principle, be explained by pre-shared randomness (i.e., a “local hidden variable) rather than anything uniquely quantum.

To see why entanglement is truly “spooky”, we now consider yet another experiment.

Experiment 2: Measurement in the Hadamard Basis. Now suppose Alice measures her qubit in the Hadamard (or “plus-minus”) basis $\{|+\rangle, |-\rangle\}$, where

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle).$$

The probability of observing outcome $j \in \{+,-\}$ is given by

$$\Pr\left[\boxed{\nearrow} = |j\rangle\right] = \left\|(|j\rangle\langle j|_A \otimes I_B)\,|\Phi^+\rangle_{AB}\right\|^2,$$

and the corresponding post-measurement state is

$$\frac{(|j\rangle\langle j|_A \otimes I_B)\,|\Phi^+\rangle_{AB}}{\sqrt{\langle\Phi^+|_{AB}\,(|i\rangle\langle i|_A \otimes I_B)\,|\Phi^+\rangle_{AB}}}.$$

Before we calculate the measurement outcomes, we first represent system A of the Bell state using the Hadamard basis, rather than the computational basis; this allows us to write

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left(\frac{|+\rangle_A + |-\rangle_A}{\sqrt{2}}\right) \otimes |0\rangle_B + \left(\frac{|+\rangle_A - |-\rangle_A}{\sqrt{2}}\right) \otimes |1\rangle_B\right).$$

Outcome $|+\rangle$: The probability that Alice obtains outcome $|+\rangle$ is

$$\begin{aligned}
\left\|(|+\rangle\langle+|_A \otimes I_B)\,|\Phi^+\rangle_{AB}\right\|^2 &= \frac{1}{2}\left\|\frac{1}{\sqrt{2}}\,|+\rangle_A \otimes |0\rangle_B + \frac{1}{\sqrt{2}}\,|+\rangle_A \otimes |1\rangle_B\right\|^2 \\
&= \frac{1}{2}\left\||+\rangle_A \otimes \left(\frac{|0\rangle_B + |1\rangle_B}{\sqrt{2}}\right)\right\|^2 \\
&= \frac{1}{2}\left\||+\rangle_A \otimes |+\rangle_B\right\|^2 \;=\; \tfrac{1}{2}.
\end{aligned}$$

Moreover, the corresponding post-measurement state is given by

$$|+\rangle_A \otimes |+\rangle_B\,.$$

Outcome $|-\rangle$: The probability that Alice obtains outcome

$$\begin{aligned}
\left\|(|-\rangle\langle-|_A \otimes I_B)\,|\Phi^+\rangle_{AB}\right\|^2 &= \frac{1}{2}\left\|\frac{1}{\sqrt{2}}\,|-\rangle_A \otimes |0\rangle_B - \frac{1}{\sqrt{2}}\,|-\rangle_A \otimes |1\rangle_B\right\|^2 \\
&= \frac{1}{2}\left\||-\rangle_A \otimes \left(\frac{|0\rangle_B - |1\rangle_B}{\sqrt{2}}\right)\right\|^2 \\
&= \frac{1}{2}\left\||-\rangle_A \otimes |-\rangle_B\right\|^2 \;=\; \tfrac{1}{2}.
\end{aligned}$$

Moreover, the corresponding post-measurement state is given by

$$|-\rangle_A \otimes |-\rangle_B\,.$$

Remarkably, Bob's state is once again perfectly correlated with Alice's, even though Alice chose a different measurement basis. This is much "spookier": somehow Bob's qubit instantaneously "knows" what Alice measured, even though they are spatially separated over a large distance.

EPR Paradox: Spooky Action at a Distance?

In their famous 1935 paper, *Einstein, Podolsky, and Rosen* argued that the phenomenon we observed in the previous two thought experiments suggests that quantum mechanics might be *incomplete*. They considered entanglement to be evidence of some underlying "hidden variables" that pre-determine outcomes. Otherwise, *how could Bob's qubit "know" what Alice measured, seemingly instantaneously?*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Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

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(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one

What bothered Einstein is that Alice's actions seem to instantaneously influence Bob's outcomes across arbitrary distances—essentially "faster than light", seemingly violating the laws of special relativity. This apparent tension between quantum mechanics and relativity is what Einstein famously called "spooky action at a distance." Later developments, such as Bell's theorem and experimental violations of Bell inequalities, showed that no local hidden variable theory can reproduce all of the predictions of quantum mechanics. We will revisit the resolution of this paradox in the next lecture.

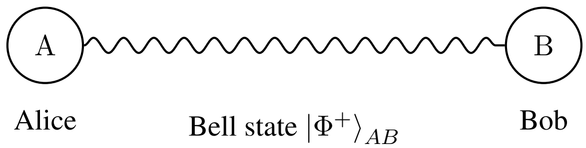
Today, quantum entanglement still remains one of the deepest and most fascinating features of quantum theory. As we will see next, it can also serve as a powerful resource for quantum protocols.

3 Quantum Teleportation

Suppose that Alice possesses a single qubit in an arbitrary state (which may or may not be known to her)

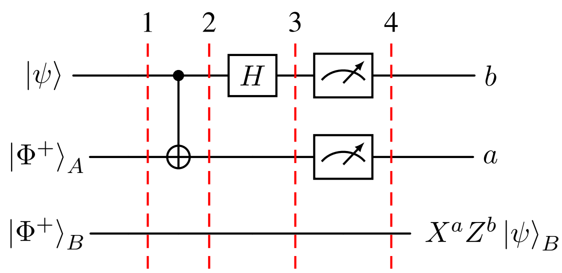
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \qquad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Her goal is to transmit this state to Bob, who is very far away. It turns out that, if Alice and Bob already share a Bell state of the form



then Alice can *teleport* the quantum information encoded in $|\psi\rangle$ directly into Bob's system using only local quantum operations and some classical communication.

To carry out the *quantum teleportation* protocol, Alice and Bob need to perform the following quantum computation, where the first two wires represent Alice's qubits, and the third wire represents Bob's qubit:



Step 1: The joint initial state. The total system consists of Alice's input qubit $|\psi\rangle$ together with the shared Bell state:

$$|\psi\rangle \otimes |\Phi^+\rangle = (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Expanding, we obtain

$$\tfrac{1}{\sqrt{2}}\big(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\big).$$

Step 2: Alice applies the CNOT gate. After the CNOT is applied, the joint state becomes

$$\frac{1}{\sqrt{2}}\big(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle\big).$$

Step 3: Alice applies a Hadamard on the first qubit. To see what happens, we do this term-by-term:

For the α-terms:

$$\alpha\,|000\rangle \xrightarrow{H} \alpha\big(H\,|0\rangle\,\big)\,|00\rangle = \alpha\frac{1}{\sqrt{2}}\big(\,|0\rangle + |1\rangle\,\big)\,|00\rangle = \frac{\alpha}{\sqrt{2}}\big(\,|000\rangle + |100\rangle\,\big),$$

$$\alpha\,|011\rangle \xrightarrow{H} \alpha\big(H\,|0\rangle\,\big)\,|11\rangle = \frac{\alpha}{\sqrt{2}}\big(\,|011\rangle + |111\rangle\,\big).$$

For the β-terms:

$$\beta\,|110\rangle \xrightarrow{H} \beta\big(H\,|1\rangle\big)\,|10\rangle = \beta\frac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big)\,|10\rangle = \frac{\beta}{\sqrt{2}}\big(\,|010\rangle - |110\rangle\,\big),$$

$$\beta\,|101\rangle \xrightarrow{H} \beta\big(H\,|1\rangle\big)\,|01\rangle = \frac{\beta}{\sqrt{2}}\big(\,|001\rangle - |101\rangle\,\big).$$

Since the state entering Step 3 carried an overall factor $1/\sqrt{2}$ from Step 1, and each expansion by H introduces another factor $1/\sqrt{2}$, the total overall prefactor is $1/2$. Collecting all terms yields

$$\frac{1}{2}\Big(\alpha\,|000\rangle + \alpha\,|100\rangle + \alpha\,|011\rangle + \alpha\,|111\rangle + \beta\,|010\rangle - \beta\,|110\rangle + \beta\,|001\rangle - \beta\,|101\rangle\Big).$$

Step 4: Measurement of Alice's qubits. Alice measures her two qubits in the computational basis, beginning with the first qubit with outcome $b \in \{0,1\}$ and the second qubit with outcome $a \in \{0,1\}$, thus

$$(b,a) \in \{0,1\} \times \{0,1\}.$$

Depending on the outcome, Bob's qubit collapses into one of four possible states, as shown below:

Alice's outcome	**Bob's qubit**	**Relation to $\|\psi\rangle$**
$(0,0)$	$\alpha\,\|0\rangle + \beta\,\|1\rangle$	$\|\psi\rangle$
$(1,0)$	$\alpha\,\|1\rangle + \beta\,\|0\rangle$	$X\,\|\psi\rangle$
$(0,1)$	$\alpha\,\|0\rangle - \beta\,\|1\rangle$	$Z\,\|\psi\rangle$
$(1,1)$	$\alpha\,\|1\rangle - \beta\,\|0\rangle$	$XZ\,\|\psi\rangle$

In other words, if Alice's measurement outcome is (b,a), then Bob's qubit in system B is of the form

$$X^a Z^b\,|\psi\rangle_B$$

where $X^a = X$, if $a = 1$, and $X^a = I$, if $a = 0$; similarly, $Z^b = Z$, if $b = 1$, and $Z^b = I$, if $b = 0$.

However, at this point of the protocol, Bob still cannot read this qubit without knowing what Alice measured—it is *scrambled*. Thus the qubit remains "encrypted" until Alice classically communicates her measurement results to Bob, who can then undo the transformations and recover the original qubit.

Step 5: Classical communication and recovery. Alice communicates the two classical bits (b,a) to Bob. With this information, Bob applies the correction $Z^b X^a$ to his qubit in system B, thereby recovering the original state $|\psi\rangle$. Importantly, Bob cannot reconstruct the state before Alice communicates her outcomes, and Alice's measurement irreversibly destroys her copy once and for all.

In conclusion, quantum teleportation achieves the remarkable transfer of an unknown quantum state using only entanglement and classical communication. Importantly, Alice's is not able to transfer information faster than light: without the use of classical communication (which is clearly susceptible to such a *cosmic speed limit*), Bob has no chance at properly recovering her state.

Experimental realizations of quantum teleportation

Quantum teleportation was originally proposed as a theoretical protocol in 1993, but it has since been realized in a wide range of experimental platforms, including photons, trapped ions, and superconducting qubits. In photonic systems, teleportation experiments have even been carried out over long distances through optical fibers and free space, demonstrating its potential for future quantum communication networks. These milestones build directly on the pioneering experimental verification of quantum entanglement, a line of work recognized by the 2022 Nobel Prize in Physics awarded to Alain Aspect, John Clauser, and Anton Zeilinger. Zeilinger, in particular, and his collaborators were among the first to demonstrate quantum teleportation of photonic states.