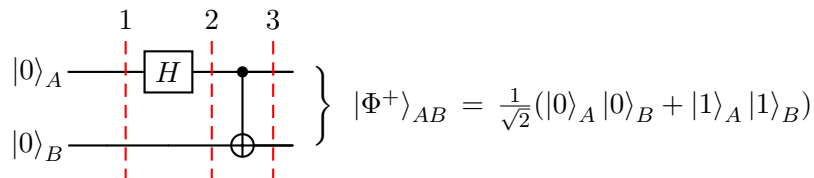


LECTURE #7: EPR PARADOX & BELL'S THEOREM

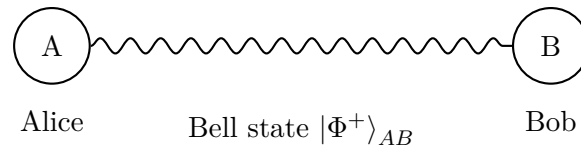
In this lecture, we will revisit the famous Einstein–Podolsky–Rosen (EPR) paradox we encountered last time. What bothered Einstein is that Alice’s actions seem to instantaneously influence Bob’s outcomes across arbitrary distances—essentially “faster than light”, seemingly violating the laws of special relativity. This apparent tension between quantum mechanics and relativity is what Einstein famously called “spooky action at a distance.” As we will see in this lecture, Einstein turned out to be wrong: Bell’s theorem and experimental violations of Bell inequalities showed that no local hidden variable theory can replicate the power of *quantum correlations*.

1 Quantum entanglement and the EPR paradox

In the last lecture, we encountered the spooky notion of quantum entanglement. In particular, we considered the following two-qubit state called the *Bell pair*:



We considered two experiments in which Alice and Bob first get together to prepare a Bell pair, and then go their separate ways across a vast distance. We looked at what happens when Alice measures in a certain basis, and how this impacted what Bob observes in his qubit.



- If Alice performs a  in the $\{|0\rangle, |1\rangle\}$ basis: $|0\rangle_A |0\rangle_B$ or $|1\rangle_A |1\rangle_B$
- If Alice performs a  in the $\{|+\rangle, |-\rangle\}$ basis: $|+\rangle_A |+\rangle_B$ or $|-\rangle_A |-\rangle_B$

So, whatever Alice does to her qubit seems to instantly affects Bob’s qubit over vast distances.

Einstein, Podolsky, Rosen (1935) argued that quantum theory suggests some sort of “faster than light communication”, which would violate the laws of special relativity. EPR argued that this is evidence that a quantum-mechanical description of nature is necessarily incomplete—it does not account for *local hidden variables* that properly explain the correlations between Alice and Bob.

2 Bell's Theorem

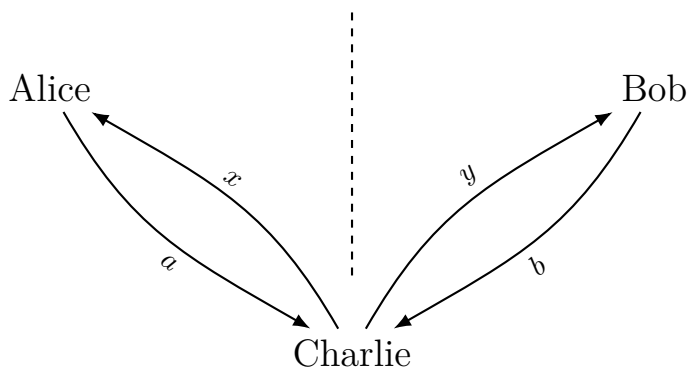
30 years later... Einstein, Podolsky and Rosen were proved wrong! Thanks to a remarkable 1964 discovery by John Bell—now known as Bell's theorem:

Bell's Theorem (1964):

No *local hidden variable theory* can explain all of the predictions of quantum theory.

John Bell put forward the idea of a *Bell test* as a means of using the power of quantum correlations to actually *falsifying* local hidden variables. A Bell test is an experiment between two cooperating and spatially separated parties Alice and Bob, and a neutral referee, say Charlie. We will consider the “CHSH game”, a more modern and simplified version of Bell's original proposal.

CHSH game. This is a game proposed by Clauser, Horne, Shimony and Holt (1969).



The CHSH game contains two separate phases: a “question phase” and an “answer phase”:

- **Questions:** Charlie samples random bits $x, y \in \{0, 1\}$ and sends them to Alice and Bob.
- **Answers:** Alice and Bob receive x and y , and independently answer with bits $a, b \in \{0, 1\}$.

Note that Alice and Bob are spatially separated and are not allowed to communicate while the game takes place. We say that Alice and Bob *win* the game, if

$$a \oplus b = x \cdot y \pmod{2}.$$

Next, we will determine the maximal winning probability p_{win} ; here, we consider all possible *strategies* employed by the cooperating parties Alice and Bob. We will show the following:

CHSH game: classical strategies

Any *classical* strategy for the CHSH game achieves a winning probability of at most $p_{\text{win}} \leq \frac{3}{4}$.

To show this, we will separately consider deterministic and randomized strategies.

- **Deterministic strategies:** Alice and Bob use deterministic strategies; meaning that their outputs are fixed functions that depend on the input bits $x, y \in \{0, 1\}$ respectively, i.e.,

Alice's answer: $a(x) \in \{0, 1\}$

Bob's answer: $b(y) \in \{0, 1\}$

In other words, a deterministic local strategy fixes two functions $a(x), b(y) \in \{0, 1\}$ ahead of time: if Alice's answer is a deterministic function of x , then so is Bob's, and his output is likewise a deterministic function of y . There are four input pairs (x, y) . Any deterministic table can satisfy the CHSH condition on at most three of these four cases; for instance:

- If $a = 0$ and $b = 0$, then $a \oplus b = 0$ and the condition holds whenever $x \cdot y = 0$, i.e., $(0, 0)$, $(0, 1)$, $(1, 0)$, but fails on $(1, 1)$. Thus $p_{\text{win}} = \frac{3}{4}$.
- If $a = x$ and $b = y$, then $a \oplus b = x \oplus y$, which equals $x \cdot y$ only for $(0, 0)$, hence $p_{\text{win}} = \frac{1}{4}$.

We can simply enumerate over all possible strategies Alice and Bob could employ, for example:

“Always send 0”

x	y	a	b	$a \oplus b$	$x \cdot y$	
0	0	0	0	0	0	✓
0	1	0	0	0	0	✓
1	0	0	0	0	0	✓
1	1	0	0	0	1	×

$$p_{\text{win}} = 0.75 = 3/4$$

“Always send 1”

x	y	a	b	$a \oplus b$	$x \cdot y$	
0	0	1	1	0	0	✓
0	1	1	1	0	0	✓
1	0	1	1	0	0	✓
1	1	1	1	0	1	×

$$p_{\text{win}} = 0.75 = 3/4$$

“Same as input” (Send back what Charlie sent)

x	y	a	b	$a \oplus b$	$x \cdot y$	
0	0	0	0	0	0	✓
0	1	0	1	1	0	×
1	0	1	0	1	0	×
1	1	1	1	0	1	×

$$p_{\text{win}} = 1/4$$

“Opposite of input” (Send back opposite of what Charlie sent)

x	y	a	b	$a \oplus b$	$x \cdot y$	
0	0	1	1	0	0	✓
0	1	1	0	1	0	×
1	0	0	1	1	0	×
1	1	0	0	0	1	×

$$p_{\text{win}} = 1/4$$

In summary, by checking all possible deterministic strategies, we can see that Alice and Bob can always be correct at most 75% of the time; in other words,

$$p_{\text{win}} \leq \frac{3}{4}.$$

- **Randomized strategies:** Alice and Bob use randomized strategies (with randomness λ).

Alice's answer: $a(x; \lambda)$

Bob's answer: $b(y; \lambda)$

Now, the function depends on some arbitrary *local hidden variable* λ , which they both have access to. Here, we imagine that λ is an arbitrary random variable, sampled from a (possibly highly complex) probability distribution. Notice that the answers a and b are no longer fixed functions of x and y respectively; instead, they are probabilistic answers that depend on λ . We will show that no matter what λ is, the winning probability is again at most $p_{\text{win}} \leq \frac{3}{4}$.

To show this, we use a simple *convexity argument* (for simplicity, we also assume that λ comes from a *discrete* probability distribution¹): by using the law of total probability

$$p_{\text{win}} = \sum_{\lambda} \Pr[\lambda] \cdot \underbrace{\Pr[\text{win} \mid \lambda]}_{\text{deterministic} \leq \frac{3}{4}} \leq \frac{3}{4} \cdot \sum_{\lambda} \Pr[\lambda] = \frac{3}{4} \cdot 1 = \frac{3}{4}.$$

In other words, the best possible winning probability is again at most $\frac{3}{4}$, and Einstein, Podolsky and Rosen would certainly agree!

Next, we consider *quantum* strategies and show the following.

CHSH game: quantum strategies

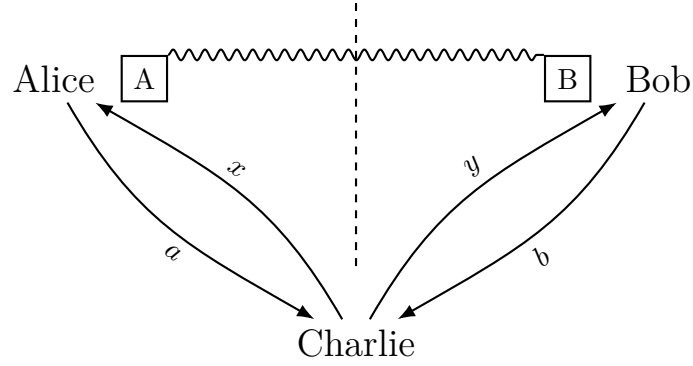
There exists a *quantum* strategy that achieves a winning probability of $p_{\text{win}} = \cos^2(\frac{\pi}{8}) \approx 0.85$.

This immediately proves Einstein wrong! So what exactly is a quantum strategy? The setup is the same as before: Alice and Bob are still unable to communicate; now, however, before the experiment starts, they get together and jointly prepare a Bell state on two systems A and B :

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$$

Next, they go their separate ways holding onto one of the qubits each. Therefore, Alice and Bob share an entangled resource: they share a *quantum* correlation rather than a classical correlation specified by some local hidden variable λ .

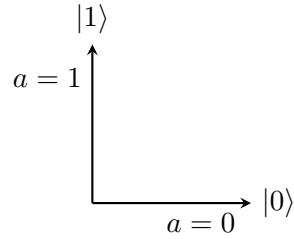
¹In general, the distribution of λ could be described by a continuous probability density.



Let us now consider a particular quantum strategy employed by Alice and Bob.

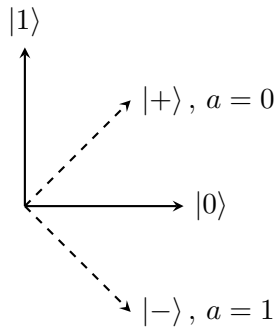
Alice's measurements.

- if $x = 0$, she measures her qubit in the A system in the $\{|0\rangle, |1\rangle\}$ basis.



If Alice measures $|0\rangle$, then she outputs 0, and if she gets the basis measurement $|1\rangle$, then she outputs 1.

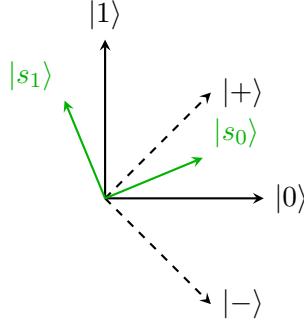
- if $x = 1$, she measures her qubit in the A system in the Hadamard basis $\{|+\rangle, |-\rangle\}$.



Now, if Alice measures $|+\rangle$, then she outputs 0, and if she gets the basis measurement $|-\rangle$, then she outputs 1.

Bob's measurements.

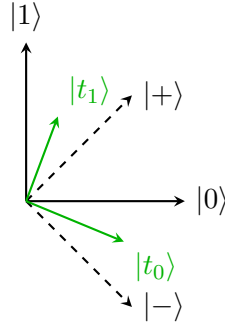
- if $y = 0$, he measures in the $\pi/8$ basis $\{|s_0\rangle, |s_1\rangle\}$:



$$\begin{aligned} |s_0\rangle &= \cos\left(\frac{\pi}{8}\right) |0\rangle + \sin\left(\frac{\pi}{8}\right) |1\rangle, \\ |s_1\rangle &= -\sin\left(\frac{\pi}{8}\right) |0\rangle + \cos\left(\frac{\pi}{8}\right) |1\rangle \end{aligned}$$

If Bob measures $|s_0\rangle$, then he outputs 0, and if he gets the basis measurement $|s_1\rangle$, then he outputs 1.

- if $y = 1$, he measures in another basis $\{|t_0\rangle, |t_1\rangle\}$ rotated by $\pi/8$:



$$\begin{aligned} |t_0\rangle &= \cos\left(\frac{\pi}{8}\right) |0\rangle - \sin\left(\frac{\pi}{8}\right) |1\rangle, \\ |t_1\rangle &= \sin\left(\frac{\pi}{8}\right) |0\rangle + \cos\left(\frac{\pi}{8}\right) |1\rangle \end{aligned}$$

If Bob measures $|t_0\rangle$, then he outputs 0, and if he gets the basis measurement $|t_1\rangle$, then he outputs 1.

Claim:

If Alice and Bob execute this particular strategy, then no matter what x and y are:

$$p_{\text{win}} = \cos^2\left(\frac{\pi}{8}\right) \approx 0.85$$

This can easily be verified by checking all possible cases. Suppose that $x = 0$, $y = 0$. Then,

- If Alice gets $|0\rangle$, then $|\phi^+\rangle_{AB} \mapsto |0\rangle_A |0\rangle_B$ and Bob outputs 0 (hence a win) with probability

$$\|(|s_0\rangle \langle s_0|) |0\rangle_B\|^2 = |\langle s_0|0\rangle|^2 = \cos^2\left(\frac{\pi}{8}\right)$$

- If Alice gets $|1\rangle$, then $|\phi^+\rangle_{AB} \mapsto |1\rangle_A |1\rangle_B$ and Bob outputs 1 (hence a win) with probability

$$\|(|s_1\rangle\langle s_1|)|1\rangle_B\|^2 = |\langle s_1|1\rangle|^2 = \cos^2(\frac{\pi}{8}).$$

Suppose $x = 1, y = 1$. Then,

- If Alice gets $|+\rangle$, she outputs 0 and $|\phi^+\rangle_{AB} \mapsto |+\rangle_A |+\rangle_B$. Bob outputs 1 (hence a win) with probability

$$\|(|t_1\rangle\langle t_1|)|+\rangle_B\|^2 = \cos^2(\frac{\pi}{8}).$$

- If Alice gets $|-\rangle$, she outputs 1 and $|\phi^+\rangle_{AB} \mapsto |-\rangle_A |-\rangle_B$ and Bob outputs 0 (hence a win) with probability

$$\|(|t_0\rangle\langle t_0|)|-\rangle_B\|^2 = \cos^2(\frac{\pi}{8}).$$

Similarly, we can check the remaining cases $x = 1, y = 0$ and $x = 0, y = 1$, where we also find that

$$p_{\text{win}} = \cos^2(\frac{\pi}{8}) \approx 0.85 > 3/4.$$

Experimental realizations of Bell tests

Bell tests are experiments designed to measure correlations between entangled particles and compare them to the limits set by Bell inequalities, which quantify the predictions of any local hidden-variable theory. These inequalities take the form of mathematical bounds on how strongly measurement outcomes can be correlated if the world were governed only by classical notions of locality and realism. A violation of these inequalities, as observed in numerous experiments, showed conclusively that nature cannot be explained by classical hidden variables, but is fundamentally quantum mechanical.

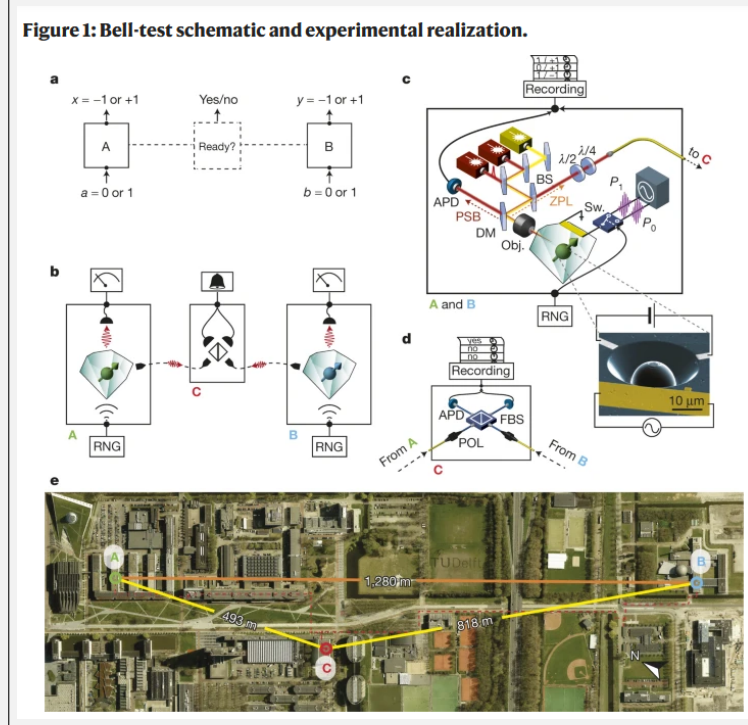


Figure credit: Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, *Nature*, volume 526, pages 682–686 (2015).