**CS 599 P1:** Introduction to Quantum Computation | Boston University
**Instructor:** Alexander Poremba | Fall 2025

# PRACTICE WORKSHEET #2

This is the second practice worksheet—it will not be graded and is meant to help you practice the tensor product notation, as well as quantum measurements. I encourage you to work through these problems by **Tuesday, September 23rd**—before the next homework assignment is out. **Solutions will be posted 09/21**.

**Background: Tensor product.** Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces, e.g. $\mathcal{H}_A = \mathbb{C}^2$ and $\mathcal{H}_B = \mathbb{C}^2$. For any $|\psi\rangle, |\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}_A$, $|\phi\rangle, |\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}_B$, and $\alpha, \beta \in \mathbb{C}$, the tensor product $\otimes$ satisfies:

- **Linearity in the first argument:**
$$(\alpha |\psi_1\rangle_A + \beta |\psi_2\rangle_A) \otimes |\phi\rangle_B = \alpha |\psi_1\rangle_A \otimes |\phi\rangle_B + \beta |\psi_2\rangle_A \otimes |\phi\rangle_B .$$
- **Linearity in the second argument:**
$$|\psi\rangle_A \otimes (\alpha |\phi_1\rangle_B + \beta |\phi_2\rangle_B) = \alpha |\psi\rangle_A \otimes |\phi_1\rangle_B + \beta |\psi\rangle_A \otimes |\phi_2\rangle_B .$$
- **Scalars factor out:**
$$(\alpha |\psi\rangle_A) \otimes |\phi\rangle_B = \alpha (|\psi\rangle_A \otimes |\phi\rangle_B).$$
- **Zero vector identity:**
$$\mathbf{0}_A \otimes |\phi\rangle_B = \mathbf{0}_{AB} = |\psi\rangle_A \otimes \mathbf{0}_B.$$

**Problem 1 (Two-qubit tensor products).** Recall that a two-qubit state lives in a tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$, which has a computational basis consisting of four basis states

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}.$$

In this practice exercise, you are asked to express each of the following set of two-qubit states as a complex linear combination of the four basis vectors above.

1. Expand the two-qubit state
$$\left(\tfrac{1}{\sqrt{2}} |0\rangle - \tfrac{i}{\sqrt{2}} |1\rangle\right) \otimes |0\rangle .$$
2. Expand the two-qubit state
$$|1\rangle \otimes \left(\tfrac{\sqrt{3}}{2} |0\rangle + \tfrac{1}{2} |1\rangle\right) .$$
3. Expand the two-qubit state
$$\left(\tfrac{1}{\sqrt{3}} |0\rangle + \tfrac{\sqrt{2}}{\sqrt{3}} |1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}} |0\rangle - \tfrac{i}{\sqrt{2}} |1\rangle\right) .$$

   *Hint*: Try to expand one qubit at a time, using linearity in the tensor product.

**Problem 2 (Two-qubit gates).** In this exercise, you will practice how to two-qubit gates on the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Specifically, you will use the following set of gates:

$$\begin{aligned} \mathrm{CNOT}\,|a,b\rangle &= |a,\, a\oplus b\rangle\,, \\ \mathrm{CZ}\,|a,b\rangle &= (-1)^{a\cdot b}\,|a,b\rangle\,, \\ \mathrm{SWAP}\,|a,b\rangle &= |b,a\rangle\,. \end{aligned}$$

For brevity, we use denote the basis of the product space $\mathbb{C}^2 \otimes \mathbb{C}^2$ via $\{|0,0\rangle\,, |0,1\rangle\,, |1,0\rangle\,, |1,1\rangle\}$.

1. Apply the CNOT gate to the state
$$\left(\frac{1}{\sqrt{2}}\,|0\rangle - \frac{i}{\sqrt{2}}\,|1\rangle\right) \otimes |0\rangle\,.$$
Expand the resulting state in the basis $\{|0,0\rangle\,, |0,1\rangle\,, |1,0\rangle\,, |1,1\rangle\}$.

2. Apply the Controlled-Z (CZ) gate to the state
$$|+\rangle \otimes |1\rangle\,, \quad \text{where } |+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$
Expand the resulting state in the computational basis.

3. Apply the SWAP gate to the state
$$\tfrac{1}{\sqrt{2}}\big(|01\rangle - i\,|10\rangle\big).$$
Expand the resulting state.

**Problem 3 (Two-qubit measurements)** In this exercise, we will consider measurements which take place in either the computational basis $\{|0\rangle\,, |1\rangle\}$ or the Hadamard basis $\{|+\rangle\,, |-\rangle\}$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Consider the following two-qubit entangled state

$$|\Psi^+\rangle \;=\; \frac{|01\rangle + |10\rangle}{\sqrt{2}}.$$

1. Measure *both* qubits in the computational basis $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$. List each outcome, its probability, and the resulting measurement operators in the form of orthogonal projectors. What is the post-measurement state for each possible outcome?

2. Measure only the *first* qubit in the Hadamard basis $\{|+\rangle\,, |-\rangle\}$, leaving the second qubit unmeasured. For each outcome "$+$" or "$-$", list its probability and the measurement operators in the form of orthogonal projectors. What does the post-measurement state look like for each outcome? What is the conditional state of the second qubit?

3. Measure the *first* qubit in the computational basis and the *second* qubit in the Hadamard basis simultaneously. For each of the four possible outcomes $\{0,+\}$, $\{0,-\}$, $\{1,+\}$, $\{1,-\}$, list their probability and the corresponding measurement operators in the form of orthogonal projectors. What is the renormalized post-measurement state for each outcome?

*Hint*: You may find it useful to expand $|\Psi^+\rangle$ by replacing the $\{|0\rangle\,, |1\rangle\}$ basis vectors of the second qubit as linear combinations of the $\{|+\rangle\,, |-\rangle\}$ basis vectors (see Homework #1).

Recall also that measuring a single qubit of a two-qubit state, in general, results in a re-normalized post-measurement state (see measurement axiom in Lecture 4).

**Problem 4 (Entangled states are not product states)** Entanglement is a fundamental feature of quantum mechanics: some quantum states of multiple qubits cannot be expressed as a product of single-qubit states. In this exercise, we show that the standard Bell pair is genuinely entangled.

Consider the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

1. Suppose (for contradiction) that $|\Phi^+\rangle$ can be written as a product of two single-qubit states,

   $$|\Phi^+\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle),$$

   with complex numbers $\alpha, \beta, \gamma, \delta$ satisfying normalization conditions. Write out the right-hand side explicitly in the computational basis.

2. By comparing coefficients of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, show that no choice of $\alpha, \beta, \gamma, \delta$ can reproduce the amplitudes of $|\Phi^+\rangle$.

3. Conclude that $|\Phi^+\rangle$ cannot be expressed as a product state, and therefore is *entangled*.

**Problem 5 (Quantum no-cloning theorem)** One of the most surprising results in quantum information is that an *arbitrary unknown* quantum state cannot be copied perfectly. This is known as the *no-cloning theorem*. It has profound implications: it protects quantum cryptography, forbids faster-than-light signaling via entanglement, and fundamentally distinguishes quantum information from classical information.

1. Assume that there exists a universal unitary operator $U$ that can clone arbitrary single-qubit states. That is, for any qubit state $|\psi\rangle$ and a fixed blank state $|0\rangle$, the unitary satisfies

   $$U\big(|\psi\rangle \otimes |0\rangle\big) = |\psi\rangle \otimes |\psi\rangle.$$

2. Check that such a unitary would correctly clone the basis states $|0\rangle$ and $|1\rangle$:

   $$U\big(|0\rangle \otimes |0\rangle\big) = |0\rangle \otimes |0\rangle, \quad U\big(|1\rangle \otimes |0\rangle\big) = |1\rangle \otimes |1\rangle.$$

3. Now consider the superposition state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Compute what $U(|+\rangle \otimes |0\rangle)$ would give, assuming *linearity* of $U$. Compare this to the desired cloned state $|+\rangle \otimes |+\rangle$.

4. Show that these two outcomes *do not yield the same state*. Explain why this means that no such unitary $U$ can exist.