

## PRACTICE WORKSHEET #4

This is the fourth and final practice worksheet—it will not be graded and is meant to help you practice the notion of the matrix trace and the symmetries of the Bell pair. I encourage you to work through these problems by **Tuesday, November 4th**—before the next homework assignment is out.

**Background (Expansion of Linear Operators).** Any linear operator  $A : \mathbb{C}^d \rightarrow \mathbb{C}^d$ , viewed as a matrix  $A \in \mathbb{C}^{d \times d}$ , can be represented in terms of its action on an orthonormal basis  $\{|i\rangle\}_{i=1}^d$  for  $\mathbb{C}^d$ .

Using the completeness relation (or, resolution of the identity)

$$I = \sum_{i=1}^d |i\rangle \langle i|,$$

we can expand  $A$  as

$$A = I A I = \left( \sum_{i=1}^d |i\rangle \langle i| \right) A \left( \sum_{j=1}^d |j\rangle \langle j| \right) = \sum_{i,j=1}^d |i\rangle \langle i| A |j\rangle \langle j| = \sum_{i,j=1}^d A_{ij} |i\rangle \langle j|,$$

where  $A_{ij} = \langle i| A |j\rangle$  are complex coefficients. Thus,  $\{|i\rangle \langle j|\}_{i,j}$  forms an orthonormal basis (with respect to the Hilbert–Schmidt inner product  $\langle \cdot, \cdot \rangle_{\text{HS}}$  over matrices) for the vector space of all  $d \times d$  complex matrices. Any operator on  $\mathbb{C}^d$  can be expressed as a linear combination of these outer-product basis elements.

**Problem 1 (Properties of the Matrix Trace).** The *trace* of a square matrix  $A \in \mathbb{C}^{d \times d}$  is defined by

$$\text{Tr}[A] = \sum_{i=1}^d A_{ii} = \sum_{i=1}^d \langle i| A |i\rangle.$$

In this problem, you will show some of its many useful properties.

1. Show that the trace is linear:

$$\text{Tr}[A + B] = \text{Tr}[A] + \text{Tr}[B], \quad \text{Tr}[cA] = c \text{Tr}[A]$$

for all complex scalars  $c$  and matrices  $A, B \in \mathbb{C}^{d \times d}$ .

2. Show that the trace is *cyclic*, i.e. for any  $A, B \in \mathbb{C}^{d \times d}$ ,

$$\text{Tr}[AB] = \text{Tr}[BA].$$

*Hint:* Expand  $A, B$  in an orthonormal basis and carefully re-arrange the matrix coefficients.

3. Show that if  $N \in \mathbb{C}^{d \times d}$  is a normal matrix (i.e.  $NN^\dagger = N^\dagger N$ ), then

$$\text{Tr}[N] = \sum_{i=1}^d \lambda_i,$$

where  $\{\lambda_i\}$  are the eigenvalues of  $N$ . *Hint:* Use the spectral theorem and cyclicity of the trace.

4. For vectors  $a, b \in \mathbb{C}^d$ , show that

$$\text{Tr}[ba^T] = a^T b.$$

5. Show that for any matrices  $A, B \in \mathbb{C}^{d \times d}$ ,

$$\text{Tr}[A \otimes B] = \text{Tr}[A] \cdot \text{Tr}[B].$$

*Hint:* Write  $A = \sum_{i,j} A_{ij} |i\rangle \langle j|$  and similarly for  $B$ , and note that  $\text{Tr}[|i\rangle \langle j|] = \delta_{ij}$ .

**Problem 2 (Ricochet property of the Bell pair).** Consider the maximally entangled Bell state

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2.$$

Show that for any  $2 \times 2$  matrix  $M$ ,

$$(I \otimes M) |\phi^+\rangle = (M^T \otimes I) |\phi^+\rangle.$$

This identity expresses the “mirror symmetry” (or “ricochet property”) of the EPR pair: an operator acting on the second subsystem is equivalent to its transpose acting on the first subsystem.

*Hint:* Expand  $|\phi^+\rangle$  in the computational basis and apply both sides explicitly to check that their components coincide.