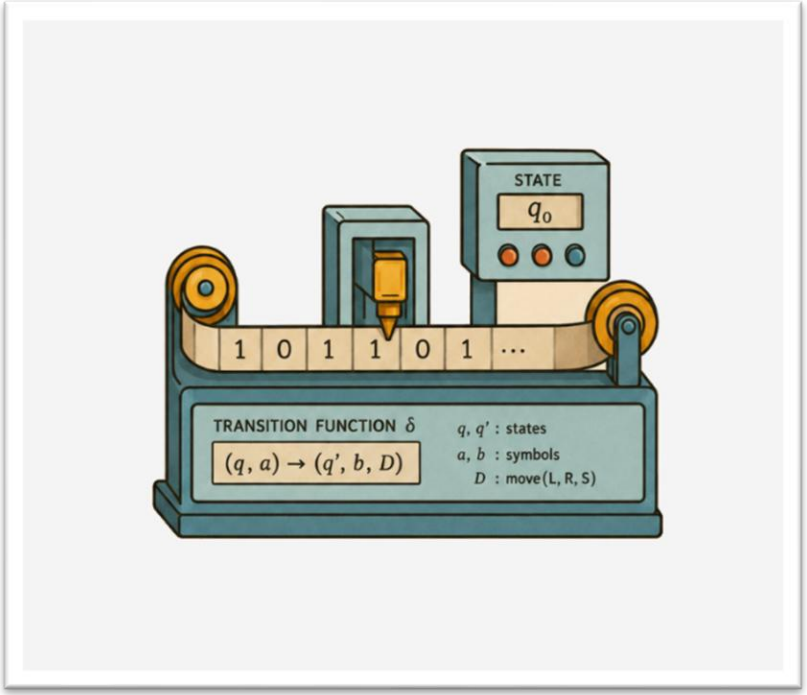
BU CS 332 – Theory of Computation

Lecture 8:

- More on non-regularity
- Test 1 review
- Quiz

Office hours:
Today 4-5.30pm, CDS 1037



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September 29, 2026

Last Time: Distinguishing Set Method

Definition: Strings x and y are **distinguishable** by L if there exists a "distinguishing extension" $z \in \Sigma^*$ such that exactly one of xz or yz is in L.

Ex: $xz \in L$ but $yz \notin L$

Definition: A set of strings S is **pairwise distinguishable** by L if every pair of distinct strings $x, y \in S$ is distinguishable by L.

Theorem: If S is pairwise distinguishable by L, then every DFA recognizing L needs at least $|S|$ states.

Corollary: If language L has an infinite pairwise distinguishable set, then L is not regular.

Palindromes

Theorem: $L = \{w \in \{0,1\}^* \mid w = w^R\}$ is not regular

Proof: We construct an infinite pairwise distinguishable set

Let $S = \{0^n \mid n \geq 0\}$.

<u>Claim</u>: S is an <u>infinite</u> pairwise distinguishable set.

Suppose $x, y \in S$ are distinct strings $(x \neq y)$.

$\Rightarrow \quad x = 0^m, \quad y = 0^n \quad (m \neq n)$

Pick $\boxed{z = 10^m}$. Then:

- $xz = 0^m \underline{10^m} \in L$
- $yz = 0^n \underline{10^m} \notin L$ because $m \neq n$.

Now it's your turn: Perfect Squares

Use the distinguishing set method to show that the following language is not regular:

$$L_2 = \left\{ 1^{n^2} \;\middle|\; n \geq 0 \right\}$$

Your job: Build an infinite set S such that for all $x \neq y \in S$, there exists a z such that exactly one of xz and yz is in L

Let $S = \{ 1^{n^2} \mid n \geq 0 \}$. Claim: S is an infinite pairw. dist. set!

$n = \sqrt{|x|}$

Given $x, y \in S$, $x \neq y$ define $z = 1^{2n+1}$. Assume $m > n$.

Then:

- $xz = 1^{n^2} \circ 1^{2n+1} = 1^{n^2+2n+1} = 1^{(n+1)^2} \in L$ ✓
- $yz = 1^{m^2} \circ 1^{2n+1} = 1^{m^2+2n+1} \notin L$

$m^2 < m^2 + 2n + 1$

$m^2 + 2n + 1 < m^2 + 2m + 1 = (m+1)^2$

Reusing a Proof

Finding a distinguishing set can take some work...

Let's try to reuse that work!

How might we show that

$\boldsymbol{BALANCED} = \{w \mid w$ has an equal # of 0s and 1s$\}$ is not regular?

$\boxed{\{0^n 1^n \mid n \geq 0\}} = \boldsymbol{BALANCED} \cap \boxed{\{w \mid \text{all 0s in } w \text{ appear before all 1s}\}}$

non-regular

regular! $L(0^* 1^*)$

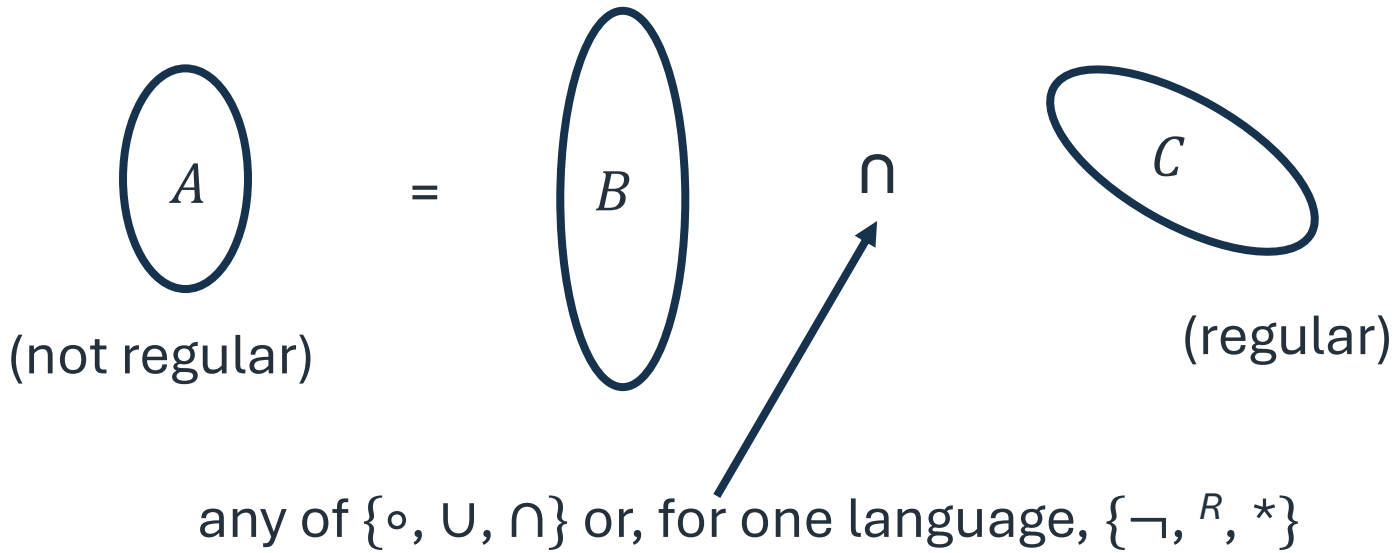
<u>Proof:</u> Suppose for the sake of contradiction that BALANCED is regular.

$\Rightarrow$ BALANCED $\cap$ $L(0^* 1^*)$ is regular (closure under intersection)

$\Rightarrow$ $\{0^n 1^n \mid n \geq 0\}$ is regular $\Rightarrow$ contradiction $\Rightarrow$ non-regular!

Using Closure Properties

If A is not regular, we can show a related language B is not regular



<u>By contradiction:</u> If B is regular, then $B \cap C$ $(= A)$ is regular.

But A is not regular so neither is B!

Example

Prove $B = \{0^i 1^j | i \neq j\}$ is not regular using

- Nonregular language

  $A = \{0^n 1^n | n \geq 0\}$ and

- Regular language

  $C = \{w \mid \text{all 0s in } w \text{ appear before all 1s}\}$

Which of the following expresses A in terms of B and C?

a) $A = B \cap C$

b) $A = \bar{B} \cap C$

c) $A = B \cup C$

d) $A = \bar{B} \cup C$

Proof that B is nonregular

Assume for the sake of contradiction that B is regular (assumption)

We know: $A = \bar{B} \cap C$

non-regular regular language $L(0^*1^*)$

$\Rightarrow$ $\bar{B}$ is regular (closure under complement)

$\Rightarrow$ $\bar{B} \cap C$ is regular (closure under intersection)

$\Rightarrow$ A is regular $\Rightarrow$ contradiction.

$\Rightarrow$ B must be non-regular.

Caution!

Let $B = \{0^i 1^j | i \neq j\}$ and write $B = A \cup C$ where

- Nonregular language

  $A = \{0^i 1^j | i > j \geq 0\}$ and
- Nonregular language

  $C = \{0^i 1^j | j > i \geq 0\}$ and

Does this let us conclude B is nonregular?

Not necessarily!

Example: $A \cup \bar{A} = \{0,1\}^*$

A: non-regular

$\bar{A}$: non-regular

$\{0,1\}^*$: regular

Test 1 Topics

Sets, Strings, Languages (0)

- Know the definition of a string and of a language (and the difference between them)
- Understand operations on strings: Concatenation, reverse
- Understand operations on languages: Union, intersection, concatenation, reverse, star, complement
- Know the difference between $\emptyset$ and ε

De Morgan's law:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Deterministic FAs (1.1)

- Given an English or formal description of a language L, draw the state diagram of a DFA recognizing L (and vice versa)
- Know the formal definition of a DFA (A DFA is a 5 tuple…) and convert between state diagram and formal description
- Know the formal definition of how a DFA computes
- Construction for closure of regular languages under complement

Nondeterministic FAs (1.2)

- Given an English or formal description of a language L, draw the state diagram of an NFA recognizing L (and vice versa)
- Know the formal definition of an NFA
- Know the subset construction for converting an NFA to a DFA
- Proving closure properties: Know the constructions for union, concatenation, star
- Know how to prove your own closure properties

Regular Expressions (1.3)

- Given an English or formal description of a language L, construct a regex generating L (and vice versa)
- Formal definition of a regex
- Know how to convert a regex to an NFA
- Know how to convert a DFA/NFA to a regex

Limitations of DFAs (Myhill-Nerode Note)

- Understand the statements of the distinguishing set method for proving DFA size lower bounds / non-regularity
- Understand the proof of why the distinguishing set method works, and be able to use it to prove similar statements
- Know how to apply the method to specific languages
- We won't ask you to prove that some language is non-regular, since you didn't have any homework problems on this yet

Test format

Problem 1: “Check your type checker”

E.g., Is aabba a string, language, or a regex?
How about {ab} U {aab}?

Problem 2: True/false with **justification**

Either provide a convincing explanation or a specific counterexample

Problems 3-5(?) Homework-style problems

Option: I Don't Know

During the exam, you can use "I don't know" for partial credit.

Option: Writing "I don't know"

For one sub-problem, you may choose to write *"I don't know"* for the problem and receive **25% of the available points** for that sub-problem. To use this option, simply write **"I don't know"** clearly below the corresponding sub-problem. Important:

- Writing "I don't know" is **mutually exclusive** with submitting a solution. You must choose one or the other. If you write any attempt at a solution, writing "I don't know" does not apply—*unless the entire attempt is clearly and completely crossed out.*
- Advice: Use this option if you do not know how to approach the problem.
- Remember: If you believe you have a promising idea, attempting the problem may earn you more than 25% through partial credit.

Study tips

- Make sure you know how to solve the problems on the practice test and are familiar with the format. The format/length of the real test will be very similar.
- If you need more practice, there are lots of problems in the book. We're happy to talk about any of these problems in office hours.
- You may bring a page of notes (writing on both sides ok) to the test. Preparing this note sheet is a great study aid.

Test tips

- You may cite without proof any result…
  - Stated in lecture
  - Stated and proved in the main body of the text (Ch. 0-1.3 + Myhill-Nerode note)
  - These include worked-out examples of state diagrams, regexes
- Not included above: homework problems, discussion problems, (solved) exercises/problems in the text
- Showing your work / explaining your answers will help us give you partial credit
- Make sure you're interpreting quantifiers (for all / there exists) correctly and in the correct order

Practice Problems

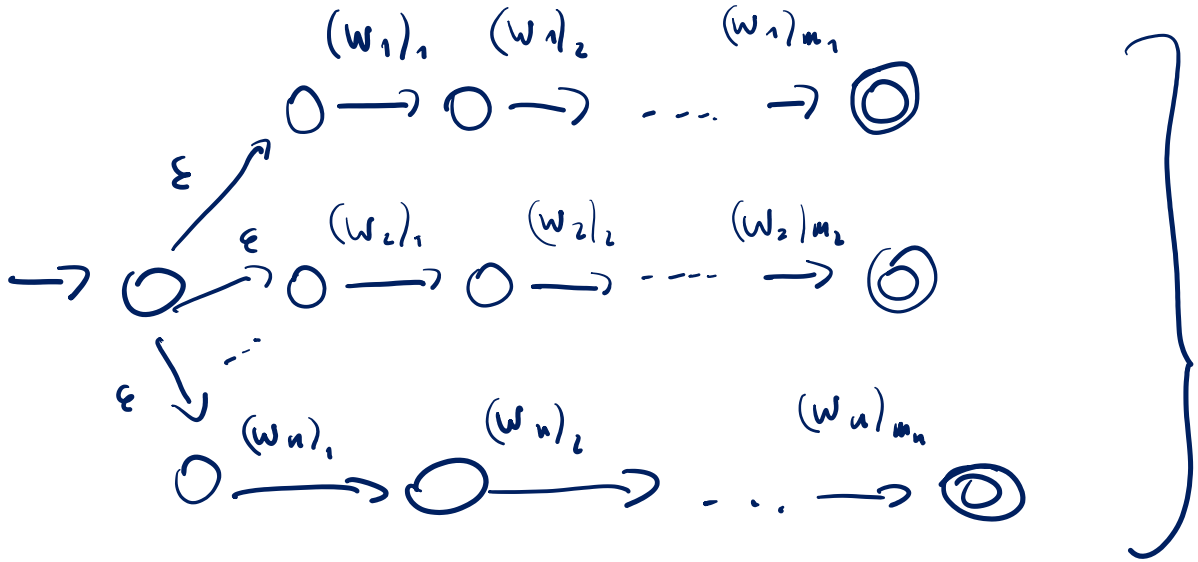
Practice Problems

Name six operations under which the regular languages are closed

Practice Problems

Prove or disprove: All finite languages are regular

<u>True</u>: L finite $\Rightarrow L = \{w_1, \ldots, w_n\}$



NFA N

N recognizes L. ✓

Practice Problems

Prove or disprove: The **non-**regular languages are closed under intersection

False: Let A be non-regular.

$$A \cap \bar{A} = \emptyset$$

non-regular

non-regular

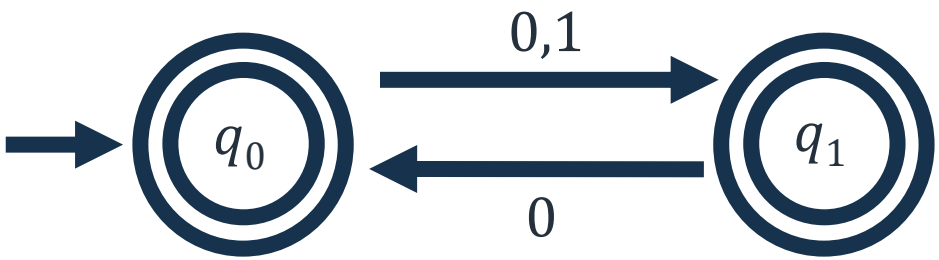
regular!

Practice Problems

Give the state diagram of an NFA recognizing the language $(01 \cup 10)^* \circ 1$

Practice Problems

Give an equivalent regular expression for the following NFA



Practice Problems

Is the following language regular?
$\{0^n 1^n \mid 0 \leq n \leq 2026\}$

Practice Problems

Is the following language regular? $\{a^n a^n | n \geq 0\}$

Quiz time!