CS 599 P1: Introduction to Quantum Computation Boston University, Fall 2025
Instructor: Alexander Poremba

Lecture # 19: Quantum Channels & Generalized Measurements

In the previous lecture, we introduced the *density matrix formalism*, a powerful generalization of the pure state formalism. In this lecture, we will expand on the density matrix formalism and introduce the corresponding measurement formalism, i.e., how to analyze various kinds of quantum measurements of a quantum system which is described by a density matrix. Lastly, we will also introduce the notion of a quantum channel—a much more general notion of how a mixed quantum state evolves. Quantum channels allow us to capture process far beyond unitary evolution; for example, quantum systems subject to noise.

Density matrix formalism. Let us briefly review the formalism from the previous lecture. A density matrix captures classical uncertainty over quantum states as well as uncertainty induced by discarding subsystems of a larger system. Given an ensemble $(\mathbf{p}, \{|\psi_j\rangle\}_{i=0}^{N-1})$ of d-dimensional quantum states with associated *probability vector* $\mathbf{p} = (p_0, p_1, \ldots, p_{N-1})$, the associated density matrix $\rho \in \mathbb{C}^{d\times d}$ is given by

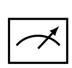
$$\rho = \sum_{j=0}^{N-1} p_j |\psi_j\rangle\langle\psi_j|.$$

where for $j \in \{0, 1, \ldots, N-1\}$ the state $|\psi_j\rangle$ occurs with probability $p_j \geq 0$ and $\sum_{j=0}^{N-1} p_j = 1$.

Every density matrix $\rho \in \mathbb{C}^{d\times d}$ satisfies the following mathematical properties:

- ρ is Hermitian ($\rho^\dagger = \rho$),
- ρ is positive semidefinite ($\rho \geq 0$),
- $\mathrm{Tr}(\rho) = 1$.

Moreover, we also saw that for any projective measurement $\{|b_i\rangle\langle b_i|\}$ (corresponding to a measurement in the orthonormal basis $\{|b_i\rangle\}$), the outcome probabilities can be expressed compactly as

$\Pr\Big[$  $= |b_i\rangle\Big] = \mathrm{Tr}\Big[|b_i\rangle\langle b_i|\rho\Big].$

We discussed why the formula illustrates the true power of the density matrix formalism: the density operator ρ alone contains all physically accessible information about the statistical ensemble.

Finally, we also saw that for a composite system AB, the *reduced density matrix* on system A is obtained via the *partial trace*,

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}).$$

This operation captures how discarding subsystem B induces classical uncertainty on A's perspective.

1 Projective measurements for density matrices

We now formalize measurements on density matrices. Projective measurements are the most familiar, corresponding to orthonormal bases and sharp eigenvalue measurements.

Projective measurement of density matrices

A *projective measurement* on a d-dimensional Hilbert space is a collection of orthogonal projectors

$$\{\Pi_i\}_{i=1}^k, \qquad \Pi_i = |b_i\rangle\langle b_i|, \qquad \Pi_i \Pi_j = \delta_{ij}\Pi_i, \qquad \sum_{i=1}^k \Pi_i = I.$$

When this measurement is applied to a density matrix ρ, the probability of observing outcome i is

$$p_i = \mathrm{Tr}(\Pi_i \rho).$$

Conditioned on outcome i, the post-measurement state is

$$\rho_i' = \frac{\Pi_i \rho \Pi_i}{\mathrm{Tr}(\Pi_i \rho)}.$$

This generalizes the familiar formula $|\langle b_i|\psi\rangle|^2$ from pure states to density matrices via the trace.

2 Generalized quantum measurements

Projective measurements, while fundamental, do not capture all quantum measurement processes. More general measurements arise naturally when a system interacts with an environment, when measurements only partially reveal information, or when detectors have multiple modes.

Generalized measurements of density matrices

A *generalized quantum measurement* is described by a collection of linear operators

$$\{M_i\}_{i=1}^k \subseteq \mathcal{L}(\mathcal{H}), \qquad \sum_{i=1}^k M_i^\dagger M_i = I\,,$$

where $\mathcal{H}$ is a d-dimensional Hilbert space. When applied to a density matrix $\rho \in \mathbb{C}^{d\times d}$:

- Outcome i occurs with probability

$$p_i = \mathrm{Tr}\big[M_i^\dagger M_i\, \rho\big].$$

- Conditioned on outcome i, the post-measurement state becomes

$$\rho_i' = \frac{M_i \rho M_i^\dagger}{\mathrm{Tr}(M_i^\dagger M_i \rho)}.$$

When each M_i is a projector P_i, we recover the definition of projective measurements.

3 POVMs: Positive-Operator Valued Measures

Recall that the notion of generalized measurements is described in terms of linear operators $E_i = M_i^\dagger M_i$. The collection $\{E_i\}$ thus conveniently contains all information about the outcome probabilities. As a final example of a quantum measurement, we now consider the notion of POVM measurements.

POVM measurement

A *Positive-Operator Valued Measure (POVM)* is a set of *positive semi-definite* operators

$$\{E_i\}_{i=1}^k, \qquad E_i \geq 0, \qquad \sum_{i=1}^k E_i = I\,,$$

acting on a d-dimensional Hilbert space. When applied to a density matrix $\rho \in \mathbb{C}^{d\times d}$, the probability of observing the outcome i is given by

$$p_i = \mathrm{Tr}\big[E_i\rho\big].$$

Every generalized measurement $\{M_i\}$ induces a POVM via $E_i = M_i^\dagger M_i$, and conversely every POVM can be realized by some generalized measurement.

POVMs will play a central role in quantum information theory. They model the most general measurement compatible with quantum mechanics.

4 Unitary evolution for density matrices

In the pure state formalism, closed-system evolution was given by

$$|\psi\rangle \longmapsto U\,|\psi\rangle\,,$$

for some unitary operator U. We now generalize this to the density matrix setting.

Recall that a pure state density matrix is $\rho = |\psi\rangle\langle\psi|$. Under unitary evolution:

$$\rho' = U\rho U^\dagger = U|\psi\rangle\langle\psi|U^\dagger,$$

which is consistent with the familiar pure-state rule. Crucially, because the mapping $\rho \mapsto U\rho U^\dagger$ is linear, unitary evolution also correctly applies to mixed states; namely, if we have

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|,$$

then, by linear, this implies that

$$U\rho U^\dagger = \sum_j p_j\, U|\psi_j\rangle\langle\psi_j|U^\dagger.$$

Thus unitary evolution preserves statistical mixtures exactly as required.

5 Quantum channels

We now generalize even further. In full generality, quantum systems may interact with unknown environments, undergo irreversible noise, or experience measurements and discarding of subsystems. The most general transformation allowed by quantum mechanics is a *quantum channel*.

Quantum channel

A *quantum channel* (or *CPTP map*) is a linear map Φ which maps linear operators over some Hilbert space $\mathcal{H}$ to linear operators over some Hilbert space $\mathcal{H}'$, i.e.,

$$\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}'),$$

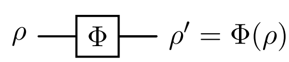
and which satisfies the following mathematical properties:

- **Completely positive**: $\Phi \otimes I_n$ is a positive operator for all n-qubit identity maps;
- **Trace-preserving**: $\mathrm{Tr}\big[\Phi(\rho)\big] = \mathrm{Tr}\big[\rho\big] = 1$, for all density operators $\rho \in \mathcal{D}(\mathcal{H})$.

Quantum channels are the mathematical objects representing all physically allowed evolutions.

5.1 Circuit representation

We depict a quantum channel acting on a state ρ as:



This emphasizes that Φ is a fully general quantum operation.

5.2 Examples of quantum channels

1. Unitary channel. The simplest example is a unitary evolution:

$$\Phi_U(\rho) = U\rho U^\dagger.$$

This is a CPTP map and preserves purity.

2. Depolarizing channel. The depolarizing channel on a d-dimensional system with parameter ε maps any state $\rho \in \mathbb{C}^{d\times d}$ toward a convex combination of the form

$$\Phi_{\text{dep}}(\rho) = (1-\varepsilon)\rho \; + \; \varepsilon\frac{I}{d}, \qquad 0 \leq \varepsilon \leq 1.$$

For $\varepsilon = 1$, all input states are mapped to the maximally mixed state.

3. Unitary evolution on a larger system, followed by a partial trace. Let $\mathcal{H}_A$ be the initial system and $\mathcal{H}_E$ an environment initially in the pure state $|0\rangle\langle 0|_E$. Then, consider the quantum channel given by

$$\Phi(\rho_A) = \mathrm{Tr}_E\Big[U_{AE}\big(\rho_A \otimes |0\rangle\langle 0|_E\big)U_{AE}^\dagger\Big].$$

This is the prototypical form of a CPTP map:

1. *Append* an environment in a fixed initial state,
2. *Apply* a global unitary U_{AE},
3. *Discard* (trace out) the environment.

It turns out that every physically realizable quantum channel can be written in this way—this is known as Stinespring's dilation. We state the theorem informally below.

Stinespring dilation

Every quantum channel Φ can be represented as

$$\Phi(\rho) = \mathrm{Tr}_E\big[U(\rho \otimes |0\rangle\langle 0|_E)U^\dagger\big]$$

for some environment system E, initial environment state $|0\rangle_E$, and unitary U.

Thus, all quantum operations can be viewed as *unitary evolution on a larger system*, followed by discarding part of the system. This theorem reveals that the appearance of *noise* or *irreversibility* is purely due to losing information to an environment.